UNIT 8 LINEAR SPACES - 1

Structure

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8.0 INTRODUCTION

Generally, the elements of a set can be arranged in any manner. It does not matter whether a three-element set is written as {a, b, c} or as {a, c, b} etc. But for certain purposes this is not helpful and we need to introduce the consideration of order. Such sets are distinguished not only by their members but also by the order in which elements appear. Thus, as an 'ordered' set, the pair (a, b) is different from the pair (b, a). An ordered set of this kind is called a 'vector'.

Here we shall consider only vectors with a finite number of elements. Further, unless stated otherwise, we shall assume that each element in a vector is a real number. Thus, we are typically concerned with the *n-tuples* $(X_1, X_2 \ldots, X_n)$ of real numbers. A vector, all of whose elements are zero, is called the ***'null vector'.***

8.1 OBJECTIVES

After going through this unit, you should be able to

- identify the presentation of vectors with magnitude and direction;
- apply the basic operations of addition, subtraction and multiplication of vectors;
- present geometrical and physical interpretation of vectors and vector operations;
- Explain the concepts of norm, inner product, vector space, linearly independent and linearly dependent set of vectors; and
- Explain the concepts of generator and basis of a vector space.

8.2 VECTOR ADDITION AND SCALAR MULTIPLICATION

A given vector, i.e., an ordered set of finite number of elements, can be written as a row, e.g., (a, b, c...) or, as a column, e.g., $\begin{pmatrix} a \\ b \\ c \\ \cdot \\ \cdot \end{pmatrix}$.

These are referred to as ***row and column vectors*** respectively.

Let us consider a vector **a** = (a_1, a_2, ..., a_n) in which each $a_i \epsilon \mathbf{R}$. Then the vector **a** is said to be a vector over the field **R**. Like ordinary numbers, ***sum and difference of two vectors*** can be defined **provided they have the same number of elements. –**

Thus, if, **a** = (a_1, a_2, ..., a_n); and **b**= (b_1,b_2, ..., b_n)

then **a + b** = (a_1+b_1, a_2+b_2, ..., a_n+b_n) and

a − b = ($a_1 - b_1$, $a_2 - b_2$, ..., $a_n - b_n$)

Note that here **a = b** if and only if $a_1 = b_1$, $a_2 = b_2$, ..., $\mathbf{a_n = b_n}$

Further, if λ is a real number, we can define ***scalar multiplication*** (λa) of vector **a** by a scalar, as follows:

(λ**a**) = (λά$_ι$, λα$_2$, ..., λά$_η$)

Example: a = (1, 3,-2), **b** = (4, 0, 6)

Then **a + b** = (1+4, 3+0, -2+6) = (5, 3, 4);

a−**b** = (1−4,3−0,-2−6,) = (-3,3,-8) and 3**a** = (3,9,-6)

Check Your Progress -1

1) If a_1 = (2,3,4,7), a_2 = (0,0,0,1) and a_3 = (1,0,1,0), find $a_1+2a_2+3a_3$.

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2) If x_1 = (2,9,8) , x_2= (0,1,0) and x_3 = (1,0,1), find $2x_2+5x_3-x_1$.

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8.3 GEOMETRICAL AND PHYSICAL INTERPRETATIONS OF VECTORS IN TWO OR THREE DIMENSION

Vector is often represented geometrically by a line with an arrowhead on it ($say, \vec{x}$). The length of the line indicates the ***magnitude of the vector,*** and the arrow denotes its ***direction***. It should be noted that a vector is not just a number. If we are considering vector lying in a plane, then two numbers are needed to describe any vector: **one for its magnitude and another giving its direction** (the angle it makes with one of the coordinate axes). If vectors in three-dimensional spaces are being studied, three numbers are needed to describe any vector: **one number for its magnitude**, and **two to denote its orientation with respect to some coordinate system**.

In general, a vector may originate at any point in space and may terminate at any point. Physical quantities such as force are, of course, independent of where one places the vector in a coordinate system. They depend only on the magnitude and direction of the vector. For this reason, it is convenient to have a vector start always at the origin of the coordinate system as in Figure 8.1. ***Here we shall, for simplicity adopt the convention that all vectors begin at the origin of the coordinate system rather than at some other point in space.***

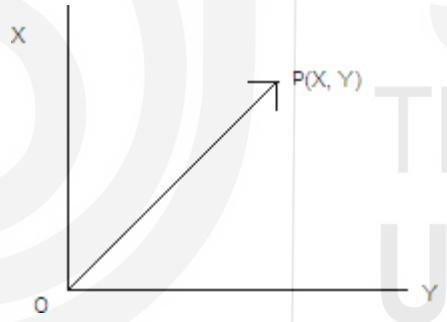

Fig.8.1: Vector starting from the origin

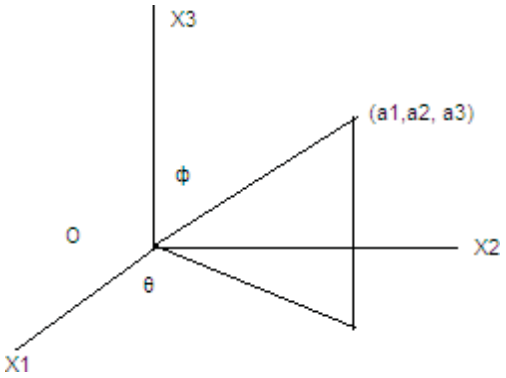

Fig.8.2: Characterised Vector

Consider a vector in Figure 8.2. It will be observed that by specifying point (a_1, a_2, a_3), that is, the point where the head of the vector terminates, we have completely characterized the vector. Its magnitude (in some physical units) is

$\sqrt{a_1^2 + a_2^2 + a_3^2}$ and its direction is characterised by the two angles $\boldsymbol{\theta}$ and $\boldsymbol{\Phi}$, where $tan\theta = \frac{a_2}{a_1}$ and $cos\phi = \frac{a_3}{\sqrt{a_1^2+a_2^2+a_3^2}}$

Thus, there is a one-to-one correspondence between all points in space and all vectors, **which emanate from the origin**. For any given point (a_1, a_2, a_3), a corresponding unique vector can be drawn from the origin to this point. Conversely, for any given vector **a**, there is a unique point (a_1, a_2, a_3), which is the point where the vector terminates. Because of this correspondence between vectors and points in 3-dimensional space, we can write **a** = (a_1, a_2, a_3).

8.4 NORM AND INNER PRODUCT

Both through scalar multiplication and vector addition transform one vector, or, a pair of vectors into ***another vector***. Unlike these concepts defined in section 8.2, we will now define functions, which transform **a pair of vectors** into a ***scalar***, a real number.

For a pair of vectors (the same length say n):

a= (a_1, a_2, …., a_n) and b = (b_1, b_2, …, b_n), we define their scalar product (as against scalar multiplication of section 8.2 as:

<**a,b**> = $a_1b_1+a_2b_2+\ldots+a_nb_n = \sum_{i=1}^{n} a_i b_i$

Scalar product is also called ***inner product*** and even ***dot product*** of two vectors, these terms are often used interchangeably to refer to the same operation.

Example: If **a** = (1, -1, 2); **b**= (-2, 3, 6),

then <**a.b**> = (1).(-2) + (11).(3) +(2) .(6) = 7

Norm of a vector: As the first step towards finding the norm of a vector, let us consider a given vector **a** = (a_1, a_2, …, a_n), and find out inner product of **a** with itself. The inner product would be $\|\boldsymbol{a}\| = \sqrt{{a_1}^2 + {a_2}^2 + \cdots + {a_n}^2}$. Now, look at the following example.

$$\boldsymbol{Example}: If\ \boldsymbol{a} = (5, -2, 3)\ then\ \|\boldsymbol{a}\| = \sqrt{5^2 + (-2)^2 + 3^2} = 38.$$

The positive square root of the inner product of a vector with itself is called the ***norm or length (or magnitude, as stated earlier) of the vector X***= (x_1,x_2,,x_n) and is denoted by ℓ**(X).**

Thus, ℓ**(X)**= $\|\boldsymbol{x}\| = [\sum_{i=1}^{n} {x_i}^2]^{\frac{1}{2}}$

In the case of the above example, the norm would be $\sqrt{38}$.

Clearly, the square of norm defined above can be regarded as the *'inner*

product' of vector **a** with itself.

Two vectors are said to be ***orthogonal*** to each other if their inner product is zero, i.e., <**a, b**>= 0,

e.g., if **a** = (1, 3), **b**=(-3,1),then <**a,b**>=0.

It is easy to see that that null vector **0** is orthogonal to every other vector.

Using the concepts of distance and the inner product, we can now define

$$\boldsymbol{cos\theta} = \frac{< \boldsymbol{a}, \boldsymbol{b} >}{\{\|\boldsymbol{a}\|.\|\boldsymbol{b}\|\}}$$

By ***Schwartz's inequality***, it is shown that $(\sum_{i=1}^{n} a_i b_1)^2 \leq \sum_{i=1}^{n} {a_i}^2 \sum_{i=1}^{n} {b_i}^2$

$\Rightarrow cos^2\theta \leq 1$ as it should be.

Two results follow right away from the above definition.

a) If $\mathbf{b} = \lambda \boldsymbol{a}\ i.e., \boldsymbol{b}\ is\ \boldsymbol{a}$ scalar multiple of a ,

then $\|b\| = |\lambda|\|a\|\ and < a, b > = \lambda\|a\| \Rightarrow cos\theta = 1\ or\ \theta = 0$

b) If <**a,b**> = 0 i.e. , **a** and **b** are orthogonal,

then $cos\theta = 0 \Longrightarrow \theta = {}^{\pi}/_{2}$. In other words, **a** and **b** are perpendicular to each other.

Example:

Find the inner product for the following vectors:

(i) **a** = (2, -1, 5) , **b** = (3, 2, 1)

(ii) **a** = (1, -1, 2) , **b** = (1, -1, -2)

Solution:

(i) <**a**, ***b***> = (2) (3) + (-1) (2) + (5) (1)

= 6-2+5=9

(ii) <**a**, ***b***> = (1) (1) +(-1) (-1) +(2) (-2)

= 1+1 – 4 = - 2

Example:

Find angle between the vectors

(i) **a** = (2, -1, 5) , **b** = (2, -1 , -1)

(ii) **a** = (3, 1, 2) , **b** = (1, 3, -3)

Solution:

(i) <**a**, **b**> = (2) (2) + (-1)(-1) +(5)(-1) = 0

$\Rightarrow cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

(ii) < **a**, **b**> = (3) (1) +(1) (3) +2(-3) = 0

$$\Rightarrow cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

8.5 VECTOR SPACES AND SUBSPACES

Let us now define a set V such that it contains, as its elements, all n-component vector that can be generated from the field of real numbers, i.e.,

$\boldsymbol{V} = \{(x_1, x_2 \dots \dots, x_n | x_i \in \mathbf{R}; i = 1,2, \dots, n\}$

Properties of vectors:

Let u,v,w be vectors and $\propto, \beta, \delta \in \mathbb{R}$**.**

1. **u+v** is a vector
2. **u+v = v+u**
3. **(u+v)+w=u+(u+w)**
4. **u+0 = 0+u =u**
   Where **0** = (0,0,…,0) is a zero vector.
5. For each **u** there exists **– u** such that **u + (-u) = (-u) + u =0**
6. $(\propto + \beta)\boldsymbol{u} = \propto \boldsymbol{u} + \beta \boldsymbol{u}$
7. $\propto (\boldsymbol{u} + \boldsymbol{v}) = \propto \boldsymbol{u} + \propto \boldsymbol{v}$
8. $(\propto \beta)\boldsymbol{u} = \propto (\beta \boldsymbol{u})$
9. $\mathbf{1}\boldsymbol{u} = \boldsymbol{u}$

The set V satisfying i, ii and iii above said to be a ***vector space or a linear space over the field*** **R**.

Summarizing: The aforesaid properties imply that V is closed under vector addition as well as under scalar multiplication of vectors. Both operations of vector addition and scalar multiplication are commutative and associative. Scalar multiplication is distributive over vector addition. And V contains an element 0, the null vector, in which each component is equal to zero.

Definition: A nonempty subset W, along with the two-operations vector addition and scalar multiplication, is said to be a **subspace of** V if

a) W is a subset of V and,

b) W is a vector space in its own right, i.e., it satisfies all the axioms specified above for a vector space.

Example:

Consider a two-dimensional plane, V_2, and suppose the set w consisting of all points lying along the horizontal (or the vertical) axis. Here w is a subset of V_2, and it satisfies all the properties listed above. In fact, all points on any line drawn through the origin constitute a subspace. Algebraically, each of the following

$W = \{(x_1, x_2) | x_1 \epsilon \mathbb{R}, x_2 = 0\}$,

$W = \{(x_1, x_2) | x_2 \in \mathbb{R}, x_1 = 0\}$

and , $W = \{(x_1, x_2) | c_1 x_1 + c_2 x_2 = 0\}$

is a subspace of $V_2 = \{(x_1, x_2) | x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\}$

Example:

Show that the vectors **a** = (1, 2, -1) , **b**= (3, 2, 5)

and **c** = (-2, 0, -6) are linearly dependent.

Solution:

Suppose $\alpha \mathbf{a} + \beta \mathbf{b} + \gamma \boldsymbol{c} = 0$

$\Rightarrow \alpha + 3\beta - 2\gamma = 0$ (1)

$2\alpha + 2\beta = 0$ (2)

$-\alpha + 5\beta - 6\gamma = 0$ (3)

From (2), $\beta = -\alpha$

From (1) , $\alpha - 3\alpha - 2\gamma \Rightarrow \gamma = -\alpha$

Thus, we may take $\alpha = 1$, $\beta = -1$, $\gamma = -1$

$\therefore$ $\boldsymbol{a}$, $\boldsymbol{b}$ *and* $\boldsymbol{c}$ are linearly dependent.

Example:

Show that any set of vectors containing zero vector is linearly dependent.

Solution:

Let $\boldsymbol{a_1} = \mathbf{0}$, $\boldsymbol{a_2}, \ldots, \boldsymbol{a_n}$ be a set of vectors.

Since $\mathbf{1}\boldsymbol{a_1} + \mathbf{0}\boldsymbol{a_2}, + \ldots, \mathbf{0}\boldsymbol{a_n} = \mathbf{0}$

we get $\boldsymbol{a_1}, \boldsymbol{a_2}, \ldots \boldsymbol{a_n}$ are linearly dependent.

8.6 LINEAR DEPENDENCE OF VECTORS

A set of vectors $\boldsymbol{x^1}, \boldsymbol{x^2}, \ldots, \boldsymbol{x^n}$ are said to be *linearly independent* if there exists scalars $c_1, c_2, \ldots, c_n$ not all zero such that $c_1\mathbf{x}^1 + c_2\mathbf{x}^2 + \ldots + c_n\mathbf{x}^n = 0$.

On the contrary, if no such scalars exist, then the vectors $\mathbf{x}^1, \ldots, \mathbf{x}^n$ are said to be linearly independent.

The term 'linearly' in the above definition is important because only linear operations, i.e., scalar multiplication and vector addition, are permitted in obtaining the null vector.

Two important results regarding linear independence or lack of it are as follows:

a) If $\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^k$ a subset of a set of vectors $\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^n$ (n > k), are linearly dependent, then the entire set is linearly dependent.

b) If a set of vectors $x^1, x^2, ...,x^n$ are linearly independent, then any subset of vectors, say, $x^1, x^2,..., x^k$ (k < n) are also linearly independent.

Example: For the two vectors x = $\begin{pmatrix}1\\0\\3\end{pmatrix}$ and y = $\begin{pmatrix}3\\0\\9\end{pmatrix}$

we have, 3**x** - **y** = 0. Therefore, the two vectors x and y are linearly dependent.

Also, from the equation, we get **y** = 3 **x**

In other words, the vector y can be obtained as scalar multiple of vector x.

In general, if k vectors $x^1, x^2, ..., x^k$ are linearly dependent, then any one of them, having non-zero coefficient in the dependence equation, can be obtained as a linear combination of the rest.

Let us consider another example:

$$x = \begin{pmatrix}1\\3\end{pmatrix}, y = \begin{pmatrix}-2\\5\end{pmatrix}, and\ z = \begin{pmatrix}3\\-2\end{pmatrix}$$

Note that x-y-z=0

Equivalently,
z = x - y

In other words, the vector **z** can be obtained as scalar multiple of the other two vectors **x and y.**

In each of these cases any of the given vectors can be obtained from the remaining vectors through scalar multiplication and vector addition (or subtraction);hence, vectors are linearly dependent.

8.7 GENERATORS AND BASIS

Consider the case of two-dimensional Euclidian space (E^2).
Note: An n-dimensional Euclidian space (Euclidian vector space) is defined as the collection of all vectors (points) **a** = (a_1,a_2, ...,a_n), for $a_I \in R$, along with the operationsof vector addition and scalar multiplication;, and with the concept of distance between the vectors.

In E^2 , for the vectors **u** = $\begin{pmatrix}1\\0\end{pmatrix}$ $and\ \boldsymbol{v} = \begin{pmatrix}0\\1\end{pmatrix}$, it is easily seen that any vector

$\boldsymbol{x} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$ may be generated from the vectors **u** and **v** as follows

$\boldsymbol{x} = x_1\boldsymbol{u} + x_2\boldsymbol{v}, for\ x_1, x_2 \in$ R.

Such vectors u and v- from which all other vectors can be obtained as linear combinations as above – are called **'generators'** of the vector space, in this case, of the two-dimensional space (E^2).

Example:

Consider the vector $\begin{pmatrix}2\\5\end{pmatrix}$. This can be expressed as:

$$2\begin{pmatrix}1\\0\end{pmatrix} + 5\begin{pmatrix}0\\1\end{pmatrix}, or\ 2\begin{pmatrix}1\\1\end{pmatrix} + 3\begin{pmatrix}0\\1\end{pmatrix},$$

$$or\ \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix} + 4\begin{pmatrix}0\\1\end{pmatrix}$$

$$or\ even\ 2\begin{pmatrix}1\\2\end{pmatrix} + 1\begin{pmatrix}0\\1\end{pmatrix} + 3\begin{pmatrix}0\\1\end{pmatrix}$$

Thus, we can generate the vector $\begin{pmatrix}2\\5\end{pmatrix}$ in a variety ways by taking a set of vectors such $\begin{pmatrix}0\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\end{pmatrix}$, **or** $\begin{pmatrix}1\\1\end{pmatrix}$, **and** $\begin{pmatrix}0\\1\end{pmatrix}, or\ \begin{pmatrix}1\\0\end{pmatrix}$, $\begin{pmatrix}1\\1\end{pmatrix}$, and $\begin{pmatrix}0\\1\end{pmatrix}$, ***or*** $\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}$and $\begin{pmatrix}0\\0\end{pmatrix}$.

Each of these pairs of these combinations of vectors like $\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}$, and $\begin{pmatrix}0\\0\end{pmatrix}$, is called a generator of E^2.

Let us discuss some important points that have to be kept in mind.

- First, an arbitrary choice of u, v etc. may not be good enough to generate all vectors $\begin{pmatrix}x_1\\x_2\end{pmatrix}$,
- Second, in the case of E^2, we need at least two vectors for this purpose. No single vector is, by itself, capable of giving us all elements of E^2. In general, for the Euclidean space E^n, we need at least n vectors for a generator.
- Third, the choice of vectors such as u, v and w can be made in a large number of alternative ways. In other words, such a set of vectors is not unique. Each of set of such vectors is said to be a ***set of generators*** for E^2.

Now, it is clear that we need at least two vectors to generate the vectors in a plane. If these two vectors are linearly independent, we can also assert that any additional vector would be unnecessary for our purpose. In fact, we need exactly two linearly independent vectors to generate all vectors $\begin{pmatrix}x_1\\x_2\end{pmatrix}$ in E^2.

Any (two) vectors of E^2, which are linearly independent and are used to generate the elements of E^2 are **said to constitute the *'basis'* of E^2.**

In general, we say that a set of vectors $a_1, a_2, ..., a_n$ from E^n ***spans or generates E^n*** if

- Every vector in E^n can be written as a linear combination of $a_1, a_2,..., a_n$.
- **Further,** if we wish to have $a_1, a_2, ..., a_n$ as a ***basis of E^n***, then additionally, we should have $a_1, a_2,...,a_n$ as a linearly independent subset of E^n.

Example:

The vector a = [2, 3, 4,] can be written uniquely in terms of the three vectors

$e_1 = [1, 0, 0]$, $e_2 = [0, 1, 0]$, and $e_3 = [0, 0, 1]$ which form a basis for

E^3: $a = 2e_1 + 3e_2 + 4e_3$

Check Your Progress -2

1) Given $x = \begin{bmatrix}5\\1\end{bmatrix}$ and $y = \begin{bmatrix}0\\3\end{bmatrix}$, find 2y and x-y graphically.

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2) Find the norm of the following vectors: (i) (2,5); (ii) (-2,2).

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3) Find a basis for the subspace
W = {(x, y, z) : 2x + y −z = 0}

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4) Find the inner product of the following pairs of vectors:

i) (2,3,4) and (4,5,5)

ii) (-2, -3, 4) and (4,5,-6).

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5) Find a basis for the subspace W= {(x, y, z): x = 2y = 3z}

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8.8 SUMMERY

This unit explained the way of presenting a variable with magnitude and direction. It also gives the idea about the basic vector operations such as addition, subtraction, and multiplication along with the concepts such as norm, inner product, generator, basis, characteristic equation, eigen value, and eigen vector. Also, the relationship between vector and matrices is covered in this unit.

8.9 ANSWER/HINTS TO CHECK YOUR PROGRESS

Answer to Check Your Progress -1

1) $a_1+2a_2+3a_3=(5,3,7,9)$
2) $2x_2+5x_3-x_1= (3,-7, -3)$

Answer to Check Your Progress -2

1) Do yourself
2) (i) 9; (ii) 8.
3) Suppose (x, y, z) ∈ W , then 2x+y −z = 0 or 2x+y
   Not that
   (x, y, z) = (x, y, 2x+y)
   = (x, 0, 2x) + (0, y, y)
   x(1, 0, 2) + y(0, 1, 1)
   Thus, W is generated by the elements (1,0,2) and (0,1,1).
   Also, note that (1,0,2) and (0,1,1) are linearly independent. If x(1,0,2) + y(0,1,1) = (0,0,0)
   then (x, y + 2x+y) = (0,0,0)
   ⟹ x=0, y=0
   Hence , (1,0,2) and (0,1,1) constitute a basis of W.

4) (i) 43; (ii) -47.
5) A basis of W is {(b,3,2)}.
   [Hint : x=2y =3z $\Longrightarrow \frac{x}{6} = \frac{y}{3} = \frac{z}{2}$]

8.9 REFERENCES AND FURTHER READINGS

1. Mathematics Class XII Part – 2 By Ravi Prakash , GRB Books, Second Edition 2025
2. BCS-012 : Basic Mathematics : https://onlinecourses.swayam2.ac.in/nou26_ma07/preview (January 2026 session)